Q2 ①

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# ASSIGNMENT-3

## ENGINEERING MATHEMATICS

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Q1 solve the differential equation

$$y'' - 4y' + 4y = 4\sin 2t$$

when $y(0) = 0$, $y'(0) = 1$ using laplace transform.

~~Assume soln to be y(t)~~
~~& $\mathcal{L}\{y(t)\} = Y(s)$ ...~~

Taking laplace transform on both sides

$\rightsquigarrow$ $y = f(t) \rightsquigarrow \mathcal{L}\{y\} = Y(s)$

$\rightsquigarrow$ $\mathcal{L}\{f^n(t)\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \dots - s^0 f^{n-1}(0)$

$\rightarrow$ $\mathcal{L}\{\sin at\} = a/s^2 + a^2$

$$\{s^2 Y - s\underbrace{y(0)}_{0} - \underbrace{y'(0)}_{1}\} - 4\{sY - \underbrace{y(0)}_{0}\} + 4Y = 4 \times 2/s^2 + 2^2$$

$$s^2 Y - 1 - 4sY + 4Y = 8/s^2 + 4$$

$$(s^2 - 4s + 4)y = \frac{8 + s^2 + 4}{s^2 + 4}$$

$$Y = \frac{s^2 + 12}{(s^2+4)(s-2)^2}$$

Making partial fractions

$$Y = \frac{As + B}{s^2 + 4} + \frac{C}{s-2} + \frac{D}{(s-2)^2}$$

$$= \frac{(As + B)\overbrace{(s-2)^2}^{s^2+4-4s} + C(s-2)(s^2+4) + D(s^2+4)}{(s^2+4)(s-2)^2}$$

$$= \frac{As^3 + 4As - 4As^2 + Bs^2 + 4B - 4Bs + Cs^3 + 4Cs - 2Cs^2 - 8C + Ds^2 + 4D}{(s^2+4)(s-2)^2}$$

$$= \frac{s^3(A+C) + s^2(-4A+B-2C+D) + s(4A-4B+4C) + 4B-8C+4D}{(s^2+4)(s-2)^2}$$ (2)

comparing coefficients.

$$A + C = 0 \quad \rightsquigarrow \quad A = 1/2$$
$$-4A + B - 2C + D = 1 \qquad B = 0$$
$$4A - 4B + 4C = 0 \qquad C = -1/2$$
$$4B - 8C + 4D = 12 \qquad D = 2.$$

$$\therefore \quad Y = \frac{\frac{1}{2}s}{s^2+4} - \frac{1}{2(s-2)} + \frac{2}{(s-2)^2}$$

Taking $\mathcal{L}^{-1}$

$$\mathcal{L}^{-1} Y = \frac{1}{2}\mathcal{L}^{-1}\frac{s}{s^2+4} - \frac{1}{2}\mathcal{L}^{-1}\frac{1}{s-2} + 2\mathcal{L}^{-1}\frac{1}{(s-2)^2}$$

$$\boxed{y(t) = \frac{1}{2}\cos 2t - \frac{1}{2}e^{2t} + 2te^{2t}.}$$

$$\left[\begin{aligned} &\text{ie } \mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2} \rightarrow \mathcal{L}^{-1}\frac{s}{s^2+a^2} = \cos at \\ &\mathcal{L}\{e^{at}\} = 1/s-a \rightsquigarrow \mathcal{L}^{-1}\, 1/s-a = e^{at}. \\ &\mathcal{L}\{t\} = \frac{\Gamma 1+1}{s^{1+1}} = \frac{1!}{s^2} \rightsquigarrow \mathcal{L}\{e^{at}t\} = \frac{1}{(s-a)^2} \\ &\therefore \mathcal{L}^{-1}\left\{\frac{1}{(s-a)^2}\right\} = e^{at}t. \end{aligned}\right]$$

Q2. Find inverse laplace transform of.

$$\frac{(s+2)^2}{(s^2+4s+8)^2}$$

$$\mathcal{L}^{-1}\left\{\frac{(s+2)^2}{(s^2+4s+8)^2}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{(s+2)^2}{(s^2+4s+4+4)^2}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{(s+2)^2}{((s+2)^2+4)^2}\right\}$$

$$= e^{-2t}\mathcal{L}^{-1}\left\{\frac{s^2}{(s^2+4)^2}\right\}$$

$$\boxed{\begin{array}{l}\text{ie } \mathcal{L}\{e^{at}f(t)\} = F(s-a).\\ \therefore \mathcal{L}^{-1}\{F(s-a)\} = e^{at}f(t).\end{array}}$$

$$= e^{-2t}\mathcal{L}^{-1}\left\{\frac{s^2+4-4}{(s^2+4)^2}\right\}$$

$$= e^{-2t}\mathcal{L}^{-1}\left\{\frac{1}{s^2+4} - \frac{4}{(s^2+4)^2}\right\}$$

By linearity,

$$= e^{-2t}\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\} - 4e^{-2t}\mathcal{L}^{-1}\left\{\frac{1}{(s^2+4)^2}\right\}$$

$$= \frac{e^{-2t}\sin 2t}{2} - \cancel{4}e^{-2t}\left\{\frac{1}{\cancel{4}}\left(\frac{\sin 2t}{4} - \frac{t\cos 2t}{2}\right)\right\}$$

$$\boxed{\text{ie } \mathcal{L}^{-1}\left\{\frac{1}{(s^2+a^2)^2}\right\} = \frac{1}{2a^3}(\sin at - at\cos at)}$$

$$= e^{-2t}\left[\frac{\sin 2t}{2} - \frac{\sin 2t}{4} - \frac{t\cos 2t}{2}\right]$$

$$= \boxed{e^{-2t}\left[\frac{\sin 2t}{4} - \frac{t\cos 2t}{2}\right]}$$

Q3. Solve the following integral-differential equation using Laplace transform method.

$$y(t) = \log t + \int_0^t e^{t-u}y(u)\,du$$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{\ln t\} = \int_0^\infty e^{-st}\ln t$$

$$\left(\Gamma m = \int_0^\infty e^{-x}x^{m-1}dx\right) \qquad \text{put } st = u \rightsquigarrow dt = \frac{du}{s}$$

$$= \int_0^\infty e^{-u}\ln\left(\frac{u}{s}\right)\frac{du}{s}$$

$$= \frac{1}{s}\int_0^\infty e^{-u}(\ln(u) - \ln(s))\,du.$$

$$= \frac{1}{s}\int_0^\infty e^{-u}\ln u\,du - \frac{\ln s}{s}\int_0^\infty e^{-u}du.$$

$$\left[\Gamma m+1 = \int_0^\infty e^{-x}x^m dx \rightarrow \Gamma m+1' = \int_0^\infty \frac{d}{dm}e^{-x}x^m dx = \int_0^\infty e^{-x}x^m \ln x\,dx\right]$$

put $m=0$,

$$\therefore \quad \frac{1}{s}\Gamma_1' - \frac{\ln s}{s}\int_0^\infty e^{-u}\,du = \boxed{\frac{1}{s}\Gamma_1' - \frac{\ln s}{s}}$$

Taking Laplace trasform of integro-differential eqn

$$\mathcal{L}\{y(t)\} = \mathcal{L}\{\ln t\} + \mathcal{L}\left\{\int_0^t e^{t-u}\,y(u)\,du\right\}$$

By convolution theorem $\mathcal{L}\left\{\int_0^t f(u)\,g(t-u)\,du\right\} = F(s)\,G(s)$

$$\therefore Y = \frac{\Gamma_1' - \ln s}{s} + Y/s.$$

$$\left(1 - \frac{1}{s}\right) Y = \frac{\Gamma_1' - \ln s}{s}$$

$$(s-1)\,Y = \Gamma_1' - \ln s$$

$$Y = \frac{\Gamma_1'}{s-1} - \frac{\ln s}{s-1}$$

Taking inverse laplace trasfrom

$$\mathcal{L}^{-1} Y = \Gamma_1' \mathcal{L}\left\{\frac{1}{s-1}\right\} - \mathcal{L}^{-1}\left\{\frac{\ln s}{s-1}\right\}$$

$$y(t) = \Gamma_1' e^t - \mathcal{L}^{-1}\left\{\frac{\ln s}{s-1}\right\}.$$

Let $f(t) = \mathcal{L}^{-1}\left\{\frac{\ln s}{s-1}\right\}$ **Doubt**

$$t f(t) = \mathcal{L}^{-1}\left\{-\frac{d}{ds}\frac{\ln s}{s-1}\right\}$$

$$= \mathcal{L}^{-1}\left\{-\frac{d}{ds}\ln s^{s-1}\right\}$$

$$= \mathcal{L}^{-1}\left\{-\frac{1}{s^{s-1}}\frac{d}{ds}s^{s-1}\right\}$$

Let $y = s^{s-1} \rightsquigarrow \ln y = (s-1)\ln s.$

$$\frac{1}{y}\frac{dy}{ds} = \frac{(s-1)}{\ln s}.$$

Q4 Solve

$$\frac{dx}{dt} + 5x - 2y = t$$

$$\frac{dy}{dt} + 2x + y = 0$$

given $x(0) = 1$ ; $y(0) = 0$

Here $x, y$ are two independent variables while $t$ is dependent

let $\mathcal{L}\{x(t)\} = X(s) \rightarrow \mathcal{L}\{x\} = X \rightarrow \mathcal{L}^{-1}\{X\} = x$

$\mathcal{L}\{y(t)\} = Y(s) \rightarrow \mathcal{L}\{y\} = Y \rightarrow \mathcal{L}^{-1}\{Y\} = y$

Taking laplace transform of first equation.

$$\mathcal{L}\{dx/dt\} + 5\mathcal{L}\{x\} - 2\mathcal{L}\{y\} = \mathcal{L}\{t\}$$

$$sX - \underbrace{x(0)}_{1} + 5X - 2Y = \frac{\Gamma 1+1}{s^{1+1}} = \frac{1}{s^2}$$

$$(s+5)X - 2Y = \frac{1}{s^2} + 1 \quad — (1)$$

Taking laplace transform of second eqn.

$$\mathcal{L}\{dy/dt\} + 2\mathcal{L}\{x\} + \mathcal{L}\{y\} = 0.$$

$$sY - \underbrace{y(0)}_{0} + 2X + Y = 0.$$

$$(s+1)Y + 2X = 0 \quad — (2)$$

$$Y = \frac{-2X}{s+1} \quad — (3) \quad ; \quad X = \frac{-(s+1)Y}{2} \quad — (4)$$

put (3) in (1)

$$(s+5)X + \frac{4X}{s+1} = \frac{1+s^2}{s^2}$$

$$X\left[\frac{(s+5)(s+1)+4}{s+1}\right] = \frac{1+s^2}{s^2}$$

$$X = \frac{(1+s^2)(s+1)}{s^2[s^2+6s+9]}$$

$$X = \frac{(1+s^2)(s+1)}{s^2[s+3]^2} = \frac{s^3+s^2+s+1}{s^2[s+3]^2}$$

Making partial fractions.

$$X = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+3} + \frac{D}{(s+3)^2}$$

$$\cancel{= A(s^3+27+9s^2+27s)s + B(s^3+27+9s^2+27s) + C(s+3)\cdot s^2 + Ds^2}$$

$$= \frac{A(s+3)^2 s + B(s+3)^2 + C(s+3)s^2 + Ds^2}{s^2(s+3)^2}$$

$$= (As+B)(s^2+9+6s) + Cs^3 + 3Cs^2 + Ds^2 \;/\; ''$$

$$= As^3 + 9As + 6As^2 + Bs^2 + 9B + 6Bs + Cs^3 + 3Cs^2 + Ds^2 \;/\; ''$$

$$= s^3(A+C) + s^2(6A+B+3C+D) + s(9A+6B) + 9B \;/\; ''$$

comparing coefficients

$$A + C = 1 \qquad \rightsquigarrow B = 1/9$$

$$6A + B + 3C + D = 1 \qquad \rightsquigarrow A = 1/27$$

$$9A + 6B = 1 \qquad \rightsquigarrow C = 26/27$$

$$9B = 1 \qquad \rightsquigarrow D = -20/9$$

$$\therefore \quad X = \frac{1}{27s} + \frac{1}{9s^2} + \frac{26}{27(s+3)} - \frac{20}{9(s+3)^2}$$

Taking inverse laplace

$$\mathcal{L}^{-1}\{X\} = \frac{1}{27}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \frac{1}{9}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + \frac{26}{27}\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\} - \frac{20}{9}\mathcal{L}^{-1}\left\{\frac{1}{(s+3)^2}\right\}$$

$$\boxed{x(t) = \frac{1}{27} + \frac{1}{9}t + \frac{26}{27}e^{-3t} - \frac{20}{9}e^{-3t}t}$$

put (9) in (1)

$$-\frac{(s+5)(s+1)Y}{2} - 2Y = \frac{1+s^2}{s^2}$$

$$Y = \frac{-4(1+s^2)}{s^2[\cancel{(s+5)(s+1)+4}]\,(s+3)^2}$$

comparing coefficients

$$A + C = 0 \qquad \rightsquigarrow B = -4/9$$

$$6A + B + 3C + D = -4 \qquad \rightsquigarrow A = 8/27$$

$$9A + 6B = 0 \qquad \rightsquigarrow C = -8/27$$

$$9B = -4 \qquad \rightsquigarrow D = -40/9.$$

$$\therefore \quad Y\cancel{(s)} = \frac{8}{27s} - \frac{4}{9s^2} - \frac{8}{27(s+3)} - \frac{40}{9(s+3)^2}$$

Taking laplace inverse.

$$\boxed{y(t) = \frac{8}{27} - \frac{4}{9}t - \frac{8}{27}e^{-3t} - \frac{40}{9}e^{-3t}t}$$

Q5. Evaluate the integral using laplace transform

$$\int_0^\infty e^{-t} \frac{\sin t}{t} dt$$

$$\mathcal{L}\{\sin t\} = 1/s^2+1 = F(s)$$

$$\mathcal{L}\{f(t)/t\} = \int_s^\infty F(s)\,ds$$

$$\mathcal{L}\{\sin t/t\} = \int_s^\infty \frac{1}{s^2+1} ds$$

$$= [\tan^{-1} s]_s^\infty$$

$$= \tan^{-1}\infty - \tan^{-1} s$$

$$= \pi/2 - \tan^{-1} s$$

$$= \cot^{-1} s$$

By definition of Laplace transform,

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

comparing, here, $s = 1$

$$\therefore \int_0^\infty e^{-t} \frac{\sin t}{t} = F(s) = \mathcal{L}\left\{\frac{\sin t}{t}\right\}_{s=1} = \cot^{-1} 1 = \boxed{\frac{\pi}{4}}$$

—— x ——

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