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# ASSIGNMENT - 4

## ENGINEERING MATHEMATICS

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B.TECH 2nd SEM
SECTION C-72.
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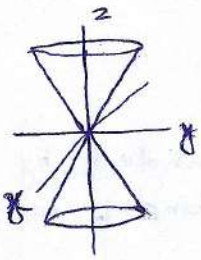
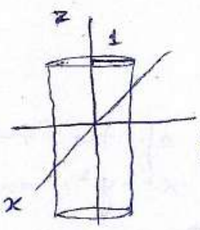
Q1. Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{1} \frac{dx\,dy\,dz}{\sqrt{x^2+y^2+z^2}}$ by changing into spherical polar coordinates system.

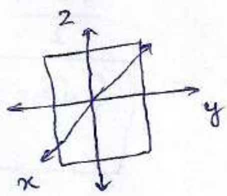
$z = \sqrt{x^2+y^2}$
$z^2 = x^2+y^2$
(cone)



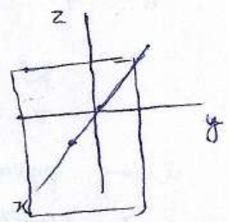

$y = \sqrt{1-x^2}$
$y^2 = 1-x^2$
$y^2+x^2 = 1$
cylinder



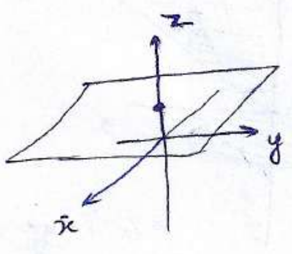

$x = 0$
(yz plane)



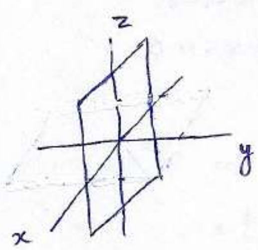

$x = 1$

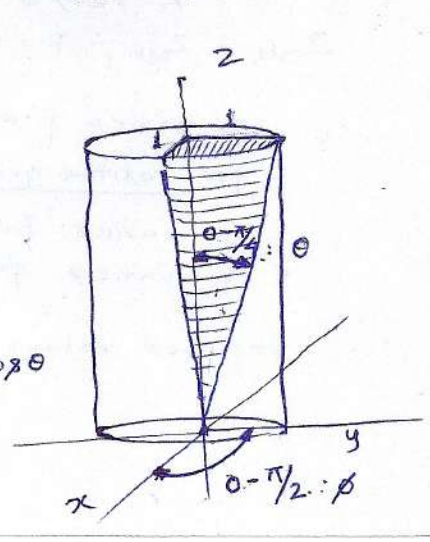



$z = 1$
(plane)




$y = 0$
(xz plane)




combining all constraints we get
in spherical coordinates where

$x = r\sin\theta\cos\phi$ ; $y = r\cos\theta\sin\phi$ , $z = r\cos\theta$

$x^2+y^2+z^2 = r^2$

$z = r\cos\theta$ if $z = 1 \rightsquigarrow 1 = r\cos\theta$

$r = \sec\theta$

$\therefore \displaystyle\int_0^1 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{1} \frac{dx\,dy\,dz}{\sqrt{x^2+y^2+z^2}}$ becomes

$$\int_0^{\pi/2} \int_0^{\pi/4} \int_0^{\sec\theta} \frac{1}{r} r^2 \sin\theta \, dr\, d\theta\, d\phi$$

$$= \int_0^{\pi/2} \int_0^{\pi/4} \left[\frac{r^2}{2}\right]_0^{\sec\theta} \sin\theta \, d\theta\, d\phi .$$

$$= \int_0^{\pi/2} \int_0^{\pi/4} \frac{\sec^2\theta}{2} \sin\theta \, d\theta\, d\phi$$

$$= \int_0^{\pi/2} \int_0^{\pi/4} \frac{1}{2} \sec\theta \tan\theta \, d\theta\, d\phi .$$

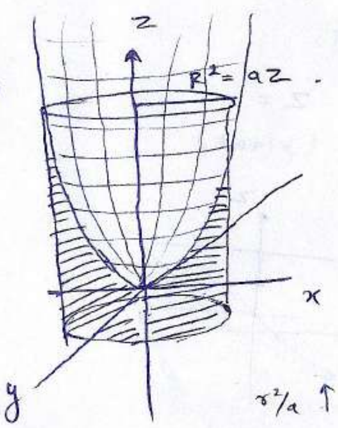
$$= \frac{1}{2} \int_0^{\pi/2} [\sec\theta]_0^{\pi/4} \, d\phi .$$

$$= \frac{1}{2} \int_0^{\pi/2} (\sqrt{2} - 1)\, d\phi .$$

$$= \frac{\sqrt{2}-1}{2} [\phi]_0^{\pi/2}$$

$$= \boxed{(\sqrt{2}-1)\frac{\pi}{4}}$$

Q2. Find the volume of the region bounded by the paraboloid $az = x^2+y^2$ and the cylinder $x^2+y^2 = R^2$, by triple integration.




$R^2 = az$ (rad. of cylinder = rad of paraboloid at intersection)

$z = R^2/a$

using $\therefore$ coordinates

$x = r\cos\theta$ ; $y = r\sin\theta$

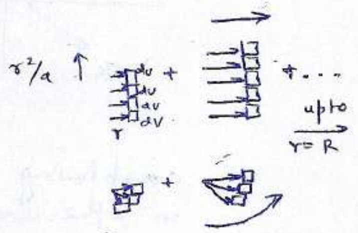
$z_{cart.} = z_{cylin.} \therefore R = r$

z varies from $0$ to $r^2/a$

for <u>positive quadrant</u>,

$\theta$ varies from $0$ to $\pi/2$

& $r$ varies from $0$ to $R$




$\therefore$ required volume $= 4 \displaystyle\int_0^{\pi/2} \int_0^{R} \int_0^{r^2/a} r\,dz\,dr\,d\theta$

$$= 4\int_0^{\pi/2}\int_0^{R} r\,[z]_0^{r^2/a}\,dr\,d\theta$$

$$= \frac{4}{a}\int_0^{\pi/2}\int_0^{R} r^3\,dr\,d\theta$$

$$= \frac{4}{a}\int_0^{\pi/2}\left[\frac{r^4}{4}\right]_0^{R} d\theta$$

$$= \frac{4}{a}\int_0^{\pi/2}\frac{R^4}{4}\,d\theta$$

$$= \frac{\cancel{4}}{a}\frac{R^4}{\cancel{4}}\,[\theta]_0^{\pi/2}$$

$$= \boxed{\frac{\pi R^4}{2a}}$$

Q3. Find the equation of enveloping cylinder of the sphere $x^2+y^2+z^2=9$ having generator parallel to the line $x/3 \pm y/2 = z/1$.

Let $x_1, y_1, z_1$ be a point on cylinder

$$\therefore \quad \frac{x-x_1}{3} = \frac{y-y_1}{2} = \frac{z-z_1}{1} = r.$$

general pt. of sphere

$$x = 3r + x_1 \; ; \; y = 2r + y_1 \; ; \; z = r + z_1$$

This pt. lies on sphere at contact.

$$\therefore \quad x^2 + y^2 + z^2 = 9$$

$$(3r + x_1)^2 + (2r + y_1)^2 + (r + z_1)^2 = 9$$

$ax^2 + bx + c = 0$ $\quad$ : $\quad x_1^2 + y_1^2 + z_1^2 + 2(3x_1 + 2y_1 + z_1)\,r + 14r^2 = 9$

Since contact point is single above eqn must give equal roots (ie same pt of contact every time)

$$b^2 = 4ac$$

$$4(3x_1 + 2y_1 + z_1)^2 = 4 \times 14\,(x_1^2 + y_1^2 + z_1^2 -$$

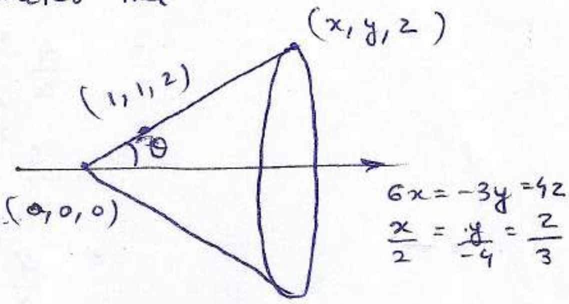
$$5x_1^2 + 10y_1^2 + 13z_1^2 + 12x_1y_1 + 4y_1z_1 + 6z_1x_1 = 126$$

locus of $x_1, y_1, z_1$ is required cylinder

$$\boxed{5x^2 + 10y^2 + 13z^2 + 12xy + 4yz + 6zx = 126}$$

Q4. Find the equation of the circular cone which passes through the point $(1, 1, 2)$ and has its vertex at the origin and axis the line $\frac{x}{2} = -\frac{y}{4} = \frac{z}{3}$

Let $x, y, z$ be a pt. on generator line.




DR of generator

$1, 1, 2$

DR of axis

$2, -4, 3$

angle b/w generator & Axis.

$$\cos\theta = \frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

$$= \frac{1 \times 2 + 1(-4) + 2(3)}{\sqrt{1^2 + 1^2 + 2^2}\sqrt{2^2 + (-4)^2 + 3^2}}$$

$$= \frac{2 - 4 + 6}{\sqrt{6}\sqrt{29}} = \frac{4}{\sqrt{174}}$$

DR of generator as per assumed

$$(x - 0, \ y - 0, \ z - 0) \equiv (x, y, z)$$

since it is the same generator, $\angle$ b/w generator and axis in both cases should be same.

$$\frac{4}{\sqrt{174}} = \frac{2x - 4y + 3z}{\sqrt{x^2 + y^2 + z^2}\sqrt{29}}$$

$$\sqrt{\frac{16 \times 29}{174}} = \frac{2x - 4y + 3z}{\sqrt{x^2 + y^2 + z^2}}$$

$$\sqrt{\frac{8}{3}}\sqrt{x^2 + y^2 + z^2} = 2x - 4y + 3z.$$

$$8(x^2 + y^2 + z^2) = 3(2x - 4y + 3z)^2$$

$$= 3(4x^2 + 16y^2 + 9z^2 - 16xy - 24yz - 12xz)$$

req. eqⁿ of cone.

$$\boxed{4x^2 + 40y^2 + 19z^2 - 48xy + 36zx - 72yz = 0.}$$

Q5. Evaluate $\iint \sqrt{\dfrac{1-x^2-y^2}{1+x^2+y^2}}\,dx\,dy$ over the positive quadrant of the circle $x^2+y^2=1$ by changing into polar coordinates

In polar coordinates $x = r\cos\theta$ ; $y = r\sin\theta$

$x^2+y^2 = r^2$ ; $dx\,dy = r\,dr\,d\theta$

$$\int_{\theta=0}^{\pi/2}\int_{r=0}^{1} \sqrt{\frac{1-r^2}{1+r^2}}\; r\,dr\,d\theta$$

$$\cancel{r = \tan\theta \sim dr = \sec^2\theta\, d\theta}$$

$$\cancel{\sqrt{\frac{1-\tan^2\phi}{1+\tan^2\phi}} \cdot \sec^2\phi =}$$

$$r^2 = \cos 2\phi \sim 2r\,dr = 2\sin 2\phi\, d\phi$$

$$\sqrt{\frac{1-r^2}{1+r^2}} = \sqrt{\frac{1-\cos 2\phi}{1+\cos 2\phi}} = \sqrt{\frac{\cancel{2}\sin^2\phi}{\cancel{2}\cos^2\phi}} = \tan\phi$$

$$= -\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{\pi/4} \tan\phi\,\sin 2\phi\; d\phi\, d\theta$$

$$= -\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{\pi/4} \frac{\sin\phi}{\cancel{\cos\phi}}\, 2\sin\phi\cancel{\cos\phi}\; d\phi\, d\theta$$

$$= -\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{\pi/4} 2\sin^2\phi\, d\phi\, d\theta$$

$$= -\int_{\theta=0}^{\pi/2}\left(\int_0^{\pi/4} (1-\cos 2\phi)\,d\phi\right) d\theta$$

$$= -\int_{\theta=0}^{\pi/2}\left[\phi - \frac{\sin 2\phi}{2}\right]_0^{\pi/4} d\theta$$

$$= -\int_{\theta=0}^{\pi/2}\left[\frac{\pi}{4} - \frac{\overset{1}{\cancel{\sin\pi/2}}}{2} - 0\right] d\theta$$

$$= -\int_{\theta=0}^{\pi/2}\left(\frac{\pi}{4} - \frac{1}{2}\right) d\theta$$

$$= -\left[\frac{\pi}{4}\theta - \frac{1}{2}\theta\right]_0^{\pi/2}$$

$$= -\left[\frac{\pi}{4}\times\frac{\pi}{2} - \frac{1}{2}\frac{\pi}{2} - 0 - 0\right]$$

$$= -\left[\frac{\pi^2}{8} - \frac{\pi}{4}\right] = \boxed{\frac{\pi}{4} - \frac{\pi^2}{8}}$$

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