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ASSIGNMENT-6
ENGINEERING MATHEMATICS

Submitted by : YASH VINAYVANSHI
B.TECH 2nd SEM
SECTION C-72
ROLL NO. 19BCS081
JAMIA MILLIA ISLAMIA
Email : yash.vinayvanshi@gmail.com

Submitted to : PROF. IDRIS QURESHI
PROF. SAIMA
PROF. NAVED AKHTAR
DEPT. OF APPLIED SCIENCES
JMI FET

Q1. Show that polar form of Cauchy - Riemann equations are,

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

Deduce that, $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$

Let (r, θ) be polar coordinate of cartesian pt (x, y)

s.t $z = x + iy = re^{i\theta}$

$\therefore$ $u + iv = f(z) = f(re^{i\theta})$ where u, v are func^n s of r, θ

Diff^n partially wrt r, θ

$$\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} = f'(re^{i\theta}) . e^{i\theta} \quad - (1)$$

$$\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = f'(re^{i\theta}) . ire^{i\theta} \quad - (2)$$

from (1) & (2)

$$\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = ir\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right) \quad - (3)$$

Equating real & imaginary parts in ep^n 3

$$\boxed{\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad ; \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}} \quad — 4 ; 5$$

CR $ep^{n}s$ in polar form

Differ^n 4 wrt r

$$\frac{\partial^2 u}{\partial r^2} = -\frac{1}{r^2}\frac{\partial v}{\partial \theta} + \frac{1}{r}\frac{\partial^2 v}{\partial \theta \partial r} \quad — 6$$

Differentiating 5 wrt θ,

$$\frac{\partial^2 u}{\partial \theta^2} = -r\frac{\partial^2 v}{\partial r \partial \theta} \quad — 7$$

using 6 $+ \frac{1}{r}$ 4 $+ \frac{1}{r^2}$ 7

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = \frac{-1}{r^2}\frac{\partial v}{\partial \theta} + \frac{1}{r}\frac{\partial^2 v}{\partial \theta \partial r} + \frac{1}{r}\left[\frac{1}{r}\frac{\partial v}{\partial \theta}\right] + \frac{1}{r^2}\left[-r\frac{\partial^2 v}{\partial r \partial \theta}\right]$$

since $\frac{\partial^2 v}{\partial r \partial \theta} = \frac{\partial^2 v}{\partial \theta \partial r}$

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = \cancel{-\frac{1}{r^2}\frac{\partial v}{\partial \theta}} + \cancel{\frac{1}{r}\frac{\partial^2 v}{\partial \theta \partial r}} + \cancel{\frac{1}{r^2}\frac{\partial v}{\partial \theta}} - \cancel{\frac{1}{r}\frac{\partial^2 v}{\partial \theta \partial r}} = 0$$

$$\therefore \boxed{\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0}$$

Q2. Determine the analytic function $f(z) = u + iv$ if $u - v = \frac{\cos x + \sin x - e^{-y}}{2(\cos x - \cosh y)}$ and $f(\pi/2) = 0$

$$f = u + iv$$
$$if = iu - v$$
$$(1+i) f = u - v + i(u+v)$$

$(1+i)\, f(z) = U + iV$ where $U = u - v$; $V = u + v$.

$$(1+i)\, f'(z) = \frac{\partial U}{\partial x} + \frac{i\,\partial V}{\partial x}$$

CR eqns

$$\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y} \quad ; \quad \frac{\partial U}{\partial y} = -\frac{\partial V}{\partial x}$$

$$(1+i)\, f'(z) = \frac{\partial U}{\partial x} - i\frac{\partial U}{\partial y}$$

$$U = \frac{\cos x + \sin x - e^{-y}}{2(\cos x - \cosh y)}$$

$$\frac{\partial U}{\partial x} = \frac{(-\sin x + \cos x)\, 2(\cos x - \cosh y) + (\cos x + \sin x - e^{-y})\, 2(\sin x)}{4(\cos x - \cosh y)^2}$$

$$\left(\frac{\partial U}{\partial x}\right)_{x=z;\, y=0} = \frac{(-\sin z + \cos z)\, 2(\cos z - 1) + (\cos z + \sin z - \ )\cdot 2\sin z}{4(\cos z - 1)^2}$$

$$= \frac{\cancel{-2\sin z\cos z} + 2\cos^2 z + \cancel{2\sin z} - 2\cos z + \cancel{2\sin z\cos z} + 2\sin^2 z \cancel{- 2\sin z}}{4(\cos z - 1)^2}$$

$$= \frac{-2\cos z + 2}{4(\cos z - 1)^2}$$

$\cancel{\cos 2\theta = 1 - 2\sin^2\theta}$

$\cancel{1 - \cos 2\theta = 2\sin^2\theta}$

$$\left(\frac{\partial U}{\partial y}\right) = \frac{e^{-y}\, 2(\cos x - \cosh y) - (\cos x + \sin x - e^{-y})(-2\sinh y)}{4(\cos x - \cosh y)^2}$$

$$\left(\frac{\partial U}{\partial y}\right)_{x=z,\, y=0} = \frac{2(\cos z - 1) - (\cos z + \sin z - 1)(0)}{4(\cos z - 1)^2}$$

$$= \frac{2(\cos z - 1)}{4(\cos z - 1)^2}$$

By milne - thomson method,

$$(1+i)\, f'(z) = \frac{-2\cos z + 2}{4(\cos z - 1)^2} - i\,\frac{2(\cos z - 1)}{4(\cos z - 1)^2}$$

$$\cancel{(1+i)}\, f'(z) = \cancel{(1+i)}\left(\frac{-\cancel{2}\cos z + \cancel{2}}{\cancel{4}(1-\cos z)^2}\right)$$

$$f'(z) = \frac{\cancel{1-\cos z}}{2(1-\cos z)^{\cancel{2}}} = \frac{1}{2(1-\cos z)}$$

$$\int f'(z) = \int \frac{1}{4\sin^2 z/2} = \int \frac{1}{4}\,\mathrm{cosec}^2\,\frac{z}{2}$$

$$f(z) = -\frac{1}{2}\cot\frac{z}{2} + c$$

$$f(\pi/2) = 0 \;\rightsquigarrow\; 0 = -\frac{1}{2}\cos\pi/4 + c \;\rightsquigarrow\; c = 1/2$$

$$\therefore \boxed{f(z) = \frac{1}{2}\left(1 - \cot\frac{z}{2}\right)}$$

Q3. Find Taylor's expansion of $f(z) = \dfrac{2z^3+1}{z^2+z}$ about the point $z = i$.

$$f(z) = \frac{2z^3+1}{z^2+z}$$

$$\begin{array}{r} 2z-2 \\ z^2+z \,\big)\, 2z^3+1 \\ -\; \pm 2z^3 + 2z^2 \\ \hline -2z^2+1 \\ -\; -2z^2 - 2z \\ + \quad\; + \\ \hline 2z+1 \end{array}$$

$$\therefore \frac{2z^3+z}{z^2+z} = 2z-2+\frac{2z+1}{z^2+z}$$

$$= 2z - 2 + \frac{2z+1}{\underset{z(z+1)}{\cancel{z^2+z}}}$$

$$= 2z - 2 + \frac{A}{z} + \frac{B}{z+1}.$$

$$= 2z - 2 + \frac{1}{z} + \frac{1}{z+1}$$

To expand $1/z$ & $1/z+1$ about $z = i$ put $z - i = t$.

$$\frac{1}{z} = \frac{1}{t+i} = \frac{1}{i}\left(1+\frac{t}{i}\right)^{-1} = \frac{1}{i}\left[1 - \frac{t}{i} + \frac{t^2}{i^2} - \frac{t^3}{i^3} + \frac{t^4}{i^4} + \dots \infty\right]$$

$i\times 1 / i\times i = -i$

$$= \cancel{\frac{1}{i}} + \frac{t}{1} + \frac{t^2}{i^3} - \frac{t^3}{i^4} + \frac{t^4}{i^5} - \dots \infty.$$

$$= -i + z - i + \sum_{n=2}^{\infty} (-1)^n \frac{(z-i)^n}{i^{n+1}}$$

& $$\frac{1}{z+1} = \frac{1}{t+i+1} = \frac{1}{1+i}\left[1+\frac{t}{1+i}\right]^{-1}$$

$$= \frac{1}{1+i}\left[1 - \frac{t}{1+i} + \frac{t^2}{(1+i)^2} - \frac{t^3}{(1+i)^3} + \frac{t^4}{(1+i)^4} - \dots\infty\right]$$

$$= \frac{1}{1+i} - \frac{t}{(1+i)^2} + \frac{t^2}{(1+i)^3} - \frac{t^3}{(1+i)^4} + \frac{t^4}{(1+i)^5} - \dots\infty$$

$$\frac{1-i}{(1+i)(1-i)} = \frac{1-i}{2} \qquad \frac{t}{1+\underset{-1}{i^2}+2i} = \frac{t}{2i}$$

$$= \frac{1+i}{2} - \frac{t}{2i} + \frac{t^2}{(i+1)^3} - \frac{t^3}{(i+1)^4} + \frac{t^4}{(i+1)^5} + \dots\infty$$

$$= \frac{1+i}{2} - \frac{z-i}{2i} + \sum_{n=2}^{\infty} \frac{(z-i)^n}{(i+1)^{n+1}}$$

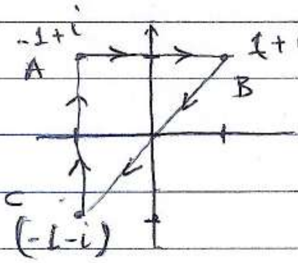
$$\therefore f(z) = 2z - 2 - i + z - i + \frac{1+i}{2} - \frac{z-i}{2i} + \sum_{n=2}^{\infty} (-1)^n \left[\frac{1}{i^{n+1}} + \frac{1}{(i+1)^{n+1}}\right](z-i)^n$$

$+ \frac{zi}{2}$

$$= 3z - \frac{3i}{2} - \frac{3}{2} - \cancel{\frac{z}{2i}} + \frac{1}{2}$$

$$\boxed{f(z) = 1 + 3z + i\left(\frac{z}{2} - \frac{3}{2}\right) + \sum_{n=2}^{\infty} (-1)^n \left[\frac{1}{(i)^{n+1}} + \frac{1}{(i+1)^{n+1}}\right](z-i)^n}$$

Qn. Verify Cauchy's theorem by integrating e^{iz} along the boundary of Δ with the vertices at pts $(1+i)$, $(-1+i)$, $(-1-i)$.



$$\int_D f(z) = \int_{AB} f(z)\,dz + \int_{BC} f(z)\,dz + \int_{CA} f(z)\,dz$$

AB : $y = 1$ $\rightsquigarrow dy = 0$; $z = x + iy$

BC : $x = y$ $\rightsquigarrow dx = dy$ $\quad dz = dx + i\,dy$

CA : $x = -1$ $\rightsquigarrow dx = 0$

$$\therefore \int_D f(z)\,dz = \int_{AB} f(z)\,dz + \int_{BC} f(z)\,dz + \int_{CA} f(z)\,dz$$

$$= \int_{AB} e^{i(x+iy)}(dx+i\,dy) + \int_{BC} e^{i(x+iy)}(dx+i\,dy)$$

$$+ \int_{CA} e^{i(x+iy)}(dx+i\,dy)$$

$$= \int_{-1}^{1} e^{i(x+i)}\,dx + \int_{1}^{-1} e^{i(x+ix)}(dx+i\,dx)$$

$$+ 0$$

$$= e^{-1}\left[e^{ix}\right]_{-1}^{1} + \left[e^{i(x+ix)}\right]_{1}^{-1} + 0$$

$$= e^{-1}\left[e^{i} - e^{-i}\right] + \left[e^{i(-1-i)} - e^{i(1+i)}\right] + 0$$

$$= e^{-1+i} - e^{-1-i} + e^{-i+1} - e^{i-1} + 0$$

$$= \boxed{0}$$

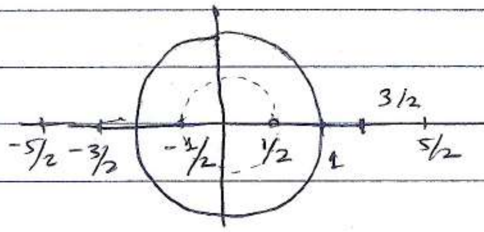
[$\because ie^{iz}$ is an analytic funcⁿ] As per cauchy's theorem, if $f(z)$ is analytic funcⁿ & $f'(z)$ is continuous ~~at~~ at each point within and on a closed curve c, then, $\int_c f(z)\,dz = 0$ which is verified above.

Q5 Evaluate $\oint_c \frac{e^z}{\cos \pi z}\,dz$, where c is the unit circle $|z| = 1$

The poles of $f(z) = \frac{e^z}{\cos \pi z}$ given by $\cos \pi z = 0$

ie at $z = \pm \frac{1}{2}, \pm \frac{3}{2}, \pm \frac{5}{2}, \ldots$

out of which only $z = \frac{1}{2}, -\frac{1}{2}$ poles.
lies inside $|z| = 1$. (simple poles)



$$\therefore \operatorname{Res} f(1/2) = \lim_{z \to 1/2} \left[z - \frac{1}{2} f(z) \right]$$

$$= \lim_{z \to 1/2} \left[\frac{(z - 1/2) e^z}{\cos \pi z} \right] \qquad \left(\frac{0}{0}\right) \quad \text{L Hospital.}$$

$$= \lim_{z \to 1/2} \left(\frac{z + ze^z - 1/2 e^z}{-\pi \sin \pi z} \right) = \frac{e^{+1/2}}{-\pi}.$$

$$\& \operatorname{Res} f(-1/2) = \lim_{z \to -1/2} \left[\frac{(z + 1/2) e^z}{\cos \pi z} \right] \qquad \left(\frac{0}{0}\right) \quad \text{L Hospital}$$

$$= \lim_{z \to -1/2} \left(\frac{\cancel{z} + e^z \cdot z + 1/2 e^z}{-\pi \sin \pi z} \right) = \frac{e^{-1/2}}{\pi}$$

$$\therefore \oint_c \frac{e^z}{\cos \pi z} dz = 2\pi i \left[\operatorname{Res} f\left(\tfrac{1}{2}\right) + \operatorname{Res} f\left(-\tfrac{1}{2}\right) \right]$$

$$= 2\cancel{\pi} i \left(\frac{-e^{+1/2}}{\cancel{\pi}} + \frac{e^{-1/2}}{\cancel{\pi}} \right)$$

$$\cancel{= 2i \left[-e^{1/2} + \frac{1}{e^{1/2}} \right]}$$

$$\cancel{= 2i \left(-e + 1 \right)}$$

$$= 2i \left[\frac{-e^{-1/2} + e^{1/2}}{2} \right] \cdot 2$$

$$= \boxed{-4i \sinh \frac{1}{2}}$$

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YASH VINAYVANSHI.